

INTEGER COMPOSITIONS WITH PART SIZES NOT EXCEEDING k

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ABSTRACT. We study the compositions of an integer n whose part sizes do not exceed a fixed integer k . We use the methods of analytic combinatorics to obtain precise asymptotic formulas for the number of such compositions, the total number of parts among all such compositions, the expected number of parts in such a composition, the total number of times a particular part size appears among all such compositions, and the expected multiplicity of a given part size in such a composition. Along the way we also obtain recurrences and generating functions for calculating several of these quantities. Our results also apply to questions about certain kinds of tilings and rhythm patterns.

1. INTRODUCTION

Let k be a fixed positive integer. We study numerical and combinatorial aspects of the sequence $\{F_n\}_{n=-\infty}^{\infty}$, defined by

$$F_n = \begin{cases} 0, & \text{if } n < 0; \\ 1, & \text{if } n = 0; \\ \sum_{a=1}^k F_{n-a}, & \text{if } n > 0, \end{cases} \quad (1.1)$$

and related sequences. When $k = 2$, the numbers F_n form the (shifted) Fibonacci sequence. The numbers F_n are known to count many things, including the tilings of the $1 \times n$ rectangle using tiles of length no more than k [2, 3], the rhythm patterns of length n with note lengths not exceeding k (Section 2, [7]), the new cells after the n th step in a budding process that starts with one cell and where each cell has one child per step, up to a total of k children per cell [15], and the compositions of an integer n with part sizes no larger than k (Section 2, [8]).

A *composition* of a positive integer n is a sequence of positive integers $(\lambda_1, \lambda_2, \dots, \lambda_m)$ such that $\sum_{j=1}^m \lambda_j = n$. The four compositions of $n = 3$, for instance, are $(1,1,1)$, $(1,2)$, $(2,1)$, and (3) . It is well known and easy to prove that a positive integer n has 2^{n-1} compositions [10]. The integers λ_j of a composition are its *parts*. If L is a set of positive integers, we can consider the set of compositions of n such that every part of each composition lies in L . Such compositions are called *restricted compositions* or *L -compositions* of n .

The literature on unrestricted integer compositions is vast, and restricted compositions have also received a fair amount of attention. For example, $\{1, 2\}$ -compositions were studied in [1], $\{1, k\}$ -compositions were studied in [4], and L -compositions in general were studied in [8]. See also [9, 10, 11, 12, 13, 16]. In addition, both restricted and unrestricted integer compositions are examples of sequence constructions in symbolic combinatorics, so several very general theorems from this field apply to them [5].

Date: February 8, 2012.

2010 Mathematics Subject Classification. 05A16, 05A15, 11B37.

Key words and phrases. Restricted composition, asymptotic, rhythm pattern, analytic combinatorics.

In this paper we fix k , let $L = \{1, 2, \dots, k\}$, and study the set of L -compositions of n , i.e., the set of compositions of n whose part sizes do not exceed k . We use the following notation.

- F_n denotes the number of $\{1, 2, \dots, k\}$ -compositions of n . (We show that this definition of F_n agrees with (1.1) in Section 2.)
- T_n denotes the total number of parts among all $\{1, 2, \dots, k\}$ -compositions of n .
- $A_n = T_n/F_n$ denotes the average number of parts of a $\{1, 2, \dots, k\}$ -composition of n .
- For $j \in \{1, 2, \dots, k\}$, $C_{n,j}$ denotes the total number of times the part size j appears among all $\{1, 2, \dots, k\}$ -compositions of n .
- For $j \in \{1, 2, \dots, k\}$, $A_{n,j} = C_{n,j}/F_n$ denotes the average number of times the part size j appears in a $\{1, 2, \dots, k\}$ -composition of n .

In this paper we study these five quantities in a unified way, with the ultimate goal of giving precise asymptotic formulas for all of them. When we say that our asymptotic formulas are *precise*, we mean that both their percentage errors and their absolute errors go to 0 as $n \rightarrow \infty$. In fact, the absolute errors for our asymptotic formulas decay at an exponential rate.

Previously, Flores [6] gave a precise asymptotic for F_n , which he derived using the theory of recurrence relations whose characteristic polynomials have no repeated roots. Then, a precise asymptotic for A_n was obtained in [5, Theorem V.1] using the methods of analytic combinatorics. Combining these immediately results in a precise asymptotic for T_n . Our asymptotic for $C_{n,j}$ is new, and our asymptotic for $A_{n,j}$ improves on the one in [5, Theorem V.2].

Along the way to developing our asymptotic formulas, we also give recursive methods for computing all five of these quantities and generating functions for F_n , T_n , and $C_{n,j}$, some of which have appeared previously. Since one of our goals is to provide a unified treatment of these five quantities, we provide proofs of all of our results, including the ones that are not new. In addition, our approach provides an alternate route to Flores's asymptotic for F_n .

We use the following notation. Let $G(x) = \sum_{a=1}^k x^a$ and let ϕ denote the unique positive real solution to

$$\frac{1}{\phi^1} + \frac{1}{\phi^2} + \dots + \frac{1}{\phi^k} = 1. \quad (1.2)$$

(That there is a unique positive real solution is guaranteed by Descartes' rule of signs.) Let G' and G'' denote the ordinary first and second derivatives of G , and let $\sigma = 1/\phi$. When $k \geq 2$ we have $\phi > 1$, and when $k = 2$, ϕ is the golden ratio. Here are the asymptotic formulas we derive in this paper. (We review big- O notation in Section 4.)

Theorem 1.1. *For some constant A with $0 < A < 1$ we have*

- (1) $F_n = \frac{\phi^{n+1}}{G'(\sigma)} + O(A^n) \quad [6],$
- (2) $T_n = \frac{\phi^{n+2}}{G'(\sigma)^2}(n+1) + \frac{\phi^{n+1}G''(\sigma)}{G'(\sigma)^3} - \frac{\phi^{n+1}}{G'(\sigma)} + O(A^n),$
- (3) $A_n = \frac{\phi}{G'(\sigma)}(n+1) - 1 + \frac{G''(\sigma)}{G'(\sigma)^2} + O(A^n) \quad [5, \text{Theorem V.1}],$
- (4) $C_{n,j} = \frac{\phi^{n+2-j}}{G'(\sigma)^2}(n+1-j) + \frac{\phi^{n+1-j}G''(\sigma)}{G'(\sigma)^3} + O(A^n), \text{ and}$
- (5) $A_{n,j} = \frac{\phi^{1-j}}{G'(\sigma)}(n+1-j) + \frac{\phi^{-j}G''(\sigma)}{G'(\sigma)^2} + O(A^n).$

Actually, the formula for A_n in Theorem 1.1 follows from an asymptotic formula for supercritical sequences, of which $\{1, 2, \dots, k\}$ -compositions are one example. See [5] for more

on supercritical sequences. Previously, the best asymptotic result we could find for $A_{n,j}$ is an asymptotic for supercritical sequences [5, Theorem V.2] which, when translated into the setting of $\{1, 2, \dots, k\}$ -compositions, becomes

$$A_{n,j} = \frac{\phi^{1-j}}{G'(\sigma)} n + O(1).$$

Our asymptotic formula for $A_{n,j}$ refines this into a precise asymptotic. Also, we remark that the formula for F_n in Theorem 1.1 was stated differently in [6], but one may use equation (1.2) to verify that the formula here is the same as the one in [6].

The rest of the paper is organized as follows. In Section 2 we review the connection between rhythm patterns and restricted compositions and we derive our recurrences. In Section 3 we use our recurrences to obtain our generating functions. We then analyze those generating functions in Section 4 to obtain our asymptotic results. Finally, Section 5 contains tables of values of F_n , T_n , A_n , $C_{n,j}$, and $A_{n,j}$ and our asymptotic approximations to them for small values of n and k .

2. RHYTHM PATTERNS AND RECURRENCES

In this section we give recursive methods for calculating the quantities F_n , T_n , A_n , $C_{n,j}$, and $A_{n,j}$. Before doing so, however, we mention that our original motivation for studying restricted compositions was musical. The L -compositions of an integer n are in bijection with the rhythm patterns of length n with note lengths in L [7]. A *rhythm pattern* of length n is a sequence of notes (also called hits) and rests played over n evenly spaced pulses, where a note occurs on the first pulse, and on each further pulse either a note or a rest (but not both) occurs. The *length* of a note is 1 plus the number of rests until the next note or the end of the pattern. The bijection is given by mapping compositions to rhythm patterns part by part, where the part j corresponds to a note of length j (i.e., a hit followed by $j - 1$ rests). For example, the composition $(2, 1, 1, 1, 2, 2, 1)$ of $n = 10$ corresponds to the rhythm pattern

hit - rest - hit - hit - hit - hit - rest - hit - rest - hit.

Thus, if we consider $\{1, 2, \dots, k\}$ -compositions, F_n counts the number of rhythm patterns of length n with no note length exceeding k , T_n counts how many notes one would play if one played all such patterns one after another, and A_n says how many notes occur on average in such a pattern. $C_{n,j}$ indicates how many times the note length j occurs among all such patterns, and $A_{n,j}$ indicates how many times the note length j occurs on average in such a pattern.

We now derive our recurrences. Our recurrence for F_n is well known, and previously appeared in [8, 15], among others.

Theorem 2.1. *F_n satisfies*

$$F_n = \begin{cases} 0, & \text{if } n < 0; \\ 1, & \text{if } n = 0; \\ \sum_{a=1}^k F_{n-a}, & \text{if } n > 0. \end{cases} \quad (2.1)$$

Proof. Certainly $F_n = 0$ if n is negative. $F_0 = 1$ as there is one composition—the empty composition—of 0. If $n > 0$ then we obtain the $\{1, 2, \dots, k\}$ -compositions of n by appending 1's to the $\{1, 2, \dots, k\}$ -compositions of $n - 1$, by appending 2's to the $\{1, 2, \dots, k\}$ -compositions of $n - 2$, etc., all the way down to appending k 's to the $\{1, 2, \dots, k\}$ -compositions of $n - k$. Thus we have $F_n = F_{n-1} + F_{n-2} + \dots + F_{n-k}$ for $n > 0$. \square

Next we obtain a recurrence for $C_{n,j}$ (and hence a recursive method for computing $A_{n,j}$).

Theorem 2.2. *Let $j \in \{1, 2, \dots, k\}$. Then*

$$C_{n,j} = \begin{cases} 0, & \text{if } n \leq 0; \\ F_{n-j} + \sum_{a=1}^k C_{n-a,j}, & \text{if } n > 0. \end{cases} \quad (2.2)$$

Proof. If $n < j$ we have $C_{n,j} = 0$, with which this recurrence agrees. Next, if $n = j$, since $F_0 = 1$ and $C_{a,j} = 0$ for $a < j$, this recurrence correctly yields $C_{n,j} = 1$.

Finally, suppose $n > j$. By induction we may assume that this recurrence correctly computes $C_{a,j}$ for all $a < n$. Split the set of $\{1, 2, \dots, k\}$ -compositions of n into two classes: those that end with j and those that do not. Among those that end with j , there are $(C_{n-j,j} + F_{n-j})$ j 's. (Specifically, there are a total of F_{n-j} j 's at the ends and a total of $C_{n-j,j}$ j 's among all other places of these compositions.) Among the $\{1, 2, \dots, k\}$ -compositions of n that end with $a \in \{1, 2, \dots, k\}$, for $a \neq j$, there are $(C_{n-a,j})$ j 's. Summing these counts across $a \in \{1, 2, \dots, k\}$ yields $C_{n,j} = F_{n-j} + \sum_{a=1}^k C_{n-a,j}$, as desired. \square

Finally we obtain a recurrence for T_n (and hence a recursive method for computing A_n).

Theorem 2.3. *T_n satisfies*

$$T_n = \begin{cases} 0, & \text{if } n \leq 0; \\ F_n + \sum_{a=1}^k T_{n-a}, & \text{if } n > 0. \end{cases} \quad (2.3)$$

Proof. $T_0 = 0$ because there are no parts in the empty composition. If $n > 0$, then we have

$$\begin{aligned} T_n &= \sum_{j=1}^k C_{n,j} \\ &= \sum_{j=1}^k \left(F_{n-j} + \sum_{a=1}^k C_{n-a,j} \right) \\ &= F_n + \sum_{a=1}^k \sum_{j=1}^k C_{n-a,j} \\ &= F_n + \sum_{a=1}^k T_{n-a}, \end{aligned}$$

using the fact that the recurrences $C_{n,j} = F_{n-j} + \sum_{a=1}^k C_{n-a,j}$ and $F_n = \sum_{j=1}^k F_{n-j}$ are both valid for $n > 0$. \square

3. GENERATING FUNCTIONS

In this section we derive generating functions for F_n , T_n , and $C_{n,j}$. Recall $G(x) = \sum_{a=1}^k x^a$. The generating function for F_n is well known, and previously appeared in [5, 8].

Theorem 3.1. *Let $F(x)$ be the ordinary generating function for the number of $\{1, 2, \dots, k\}$ -compositions of n . That is, $F(x) = F_0 + F_1x + F_2x^2 + \dots$. Then*

$$F(x) = \frac{1}{1 - G(x)}.$$

Proof. We prove this directly. Using $F_0 = 1$ and the recurrence in (2.1) for $n > 0$, we have

$$\begin{aligned}
 F(x) &= 1 + \sum_{n=1}^{\infty} F_n x^n \\
 &= 1 + \sum_{n=1}^{\infty} \sum_{a=1}^k F_{n-a} x^n \\
 &= 1 + \sum_{a=1}^k \sum_{n=1}^{\infty} F_{n-a} x^n \\
 &= 1 + \sum_{a=1}^k x^a \sum_{n=1}^{\infty} F_{n-a} x^{n-a} \\
 &= 1 + \sum_{a=1}^k x^a F(x).
 \end{aligned}$$

Solving for $F(x)$ yields $F(x) = 1/(1 - \sum_{a=1}^k x^a)$, as desired. \square

Next, for $j \in \{1, 2, \dots, k\}$, we obtain the generating function for $C_{n,j}$. We remark that this generating function can also be obtained by differentiating formula (7) on p. 293 of [5] with respect to u and evaluating at $u = 1$. There the appropriate generating function was built directly from the combinatorics of sequences. Here we derive it from our recursive formula (2.2) instead.

Theorem 3.2. *Let $j \in \{1, 2, \dots, k\}$ and let $C_j(x)$ be the ordinary generating function for the total number of occurrences of the part size j among all $\{1, 2, \dots, k\}$ -compositions of n . That is, $C_j(x) = C_{0,j} + C_{1,j}x + C_{2,j}x^2 + \dots$. Then*

$$C_j(x) = \frac{x^j}{(1 - G(x))^2}.$$

Proof. Using $C_{0,j} = 0$ and equation (2.2) we have

$$\begin{aligned}
 C_j(x) &= \sum_{n=1}^{\infty} C_{n,j} x^n \\
 &= \sum_{n=1}^{\infty} \left(F_{n-j} + \sum_{a=1}^k C_{n-a,j} \right) x^n \\
 &= \sum_{n=1}^{\infty} F_{n-j} x^n + \sum_{n=1}^{\infty} \sum_{a=1}^k C_{n-a,j} x^n \\
 &= x^j \sum_{n=1}^{\infty} F_{n-j} x^{n-j} + \sum_{a=1}^k x^a \sum_{n=1}^{\infty} C_{n-a,j} x^{n-a} \\
 &= x^j F(x) + \sum_{a=1}^k x^a C_j(x).
 \end{aligned}$$

Solving for $C_j(x)$ we obtain $C_j(x) = x^j F(x)/(1 - \sum_{a=1}^k x^a) = x^j/(1 - \sum_{a=1}^k x^a)^2$, as desired. \square

Theorem 3.2 also reveals the nice fact that for a fixed value of k , the sequences $\{C_{n,j}\}_{n=0}^{\infty}$ for $j \in \{1, 2, \dots, k\}$ are all just shifts of the same sequence. See Table 1, for example.

Finally we obtain the generating function for T_n . This generating function previously appeared more generally in the context of sequence constructions on p. 178 of [5]. Here we derive it using our recursive formula (2.3) instead.

Theorem 3.3. *Let $T(x)$ be the ordinary generating function for the total number of parts among all $\{1, 2, \dots, k\}$ -compositions of n . That is, $T(x) = T_0 + T_1x + T_2x^2 + \dots$. Then*

$$T(x) = \frac{1}{(1 - G(x))^2} - \frac{1}{1 - G(x)}.$$

Proof. Using $T_0 = 0$, the recurrence in (2.3), and Theorem 3.1, we have

$$\begin{aligned} T(x) &= \sum_{n=1}^{\infty} T_n x^n \\ &= \sum_{n=1}^{\infty} \left(F_n + \sum_{a=1}^k T_{n-a} \right) x^n \\ &= \sum_{n=1}^{\infty} F_n x^n + \sum_{a=1}^k \sum_{n=1}^{\infty} T_{n-a} x^n \\ &= F(x) - 1 + \sum_{a=1}^k x^a \sum_{n=1}^{\infty} T_{n-a} x^{n-a} \\ &= \frac{1}{1 - G(x)} - 1 + \sum_{a=1}^k x^a T(x). \end{aligned}$$

Solving for $T(x)$ yields $T(x) = 1/(1 - G(x))^2 - 1/(1 - G(x))$, as desired. \square

4. ASYMPTOTICS

In this section we use the analytic properties of our generating functions to obtain our asymptotic formulas for F_n , T_n , A_n , $C_{n,j}$, and $A_{n,j}$. In this section we use n to stand for a nonnegative integer, x a real number, and z a complex number. In addition, for purposes of nondegeneracy in our arguments, in this section we assume $k > 1$. However, it is easily verified by inspection that all of our asymptotic results also hold for $k = 1$. In Section 4.1 we review the ideas from analytic combinatorics and establish several facts we need to derive our asymptotic results. In Section 4.2 we prove our asymptotic results.

4.1. Preliminaries. We are going to obtain our asymptotic formulas using the methods of analytic combinatorics [5, 17]. Namely, we will view the generating functions obtained in Section 3 as functions of a complex variable and we will analyze their dominant singularities to give us information about the growth rate of their coefficients. First, we quickly review big- O notation.

Definition 4.1. *Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and let $f, g : \mathbb{N} \rightarrow \mathbb{R}$. We write $f(n) = O(g(n))$ if there are positive real numbers C and N for which*

$$|f(n)| \leq C|g(n)|$$

whenever $n > N$.

If f, g , and h are functions, whenever we write an equation like

$$f(n) = h(n) + O(g(n)),$$

what we mean is

$$f(n) = h(n) + E(n)$$

for some function E , where $E(n) = O(g(n))$.

The basic observation from complex analysis that we will need is this [17, Theorem 2.4.3].

Theorem 4.2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in a region containing the origin, and let $z_0 \neq 0$ be any singularity of $f(z)$ of smallest modulus. Let $R = |z_0|$ and fix a value $\epsilon > 0$. Then there exists N such that, for all $n > N$, we have*

$$|a_n| < \left(\frac{1}{R} + \epsilon \right)^n.$$

Also, for infinitely many n we have

$$|a_n| > \left(\frac{1}{R} - \epsilon \right)^n.$$

The main idea is then the following (see also [5, 17]). Let $H(z) = h_0 + h_1 z + h_2 z^2 + \dots$ be a generating function (in particular, $h_i \geq 0$ for all i) with only isolated singularities and whose coefficient growth rate we'd like to understand. We use the standard notation $[z^n]H(z)$ to stand for the coefficient h_n of z^n in $H(z)$. Suppose that, as a function of a complex variable z , $H(z)$ is analytic in a region containing the origin and has radius of convergence R . Then $H(z)$ has a singularity on the circle $\{z \in \mathbb{C} : |z| = R\}$. Suppose there is only one singularity z_0 on this circle (as there will be for our generating functions), and let $S(z)$ be the principal part of $H(z)$ at that singularity. We call z_0 the *dominant* singularity of $H(z)$. Then the function $H(z) - S(z)$ is analytic in a disk of radius $R' > R$ centered at the origin. Thus, for any fixed $\epsilon > 0$, the coefficients of its expansion at the origin cannot grow faster than

$$\left(\frac{1}{R'} + \epsilon \right)^n$$

for sufficiently large n , and hence

$$[z^n]H(z) = [z^n]S(z) + O\left(\left(\frac{1}{R'} + \epsilon\right)^n\right).$$

This is especially good when $R' > 1$: In this case we obtain that $[z^n]S(z)$ is a precise asymptotic for h_n . (In particular, we obtain that $[z^n]H(z) = [z^n]S(z) + O(A^n)$ for some number A with $0 < A < 1$.) We remark that even when it is not the case that $R' > 1$, one can still use this formulation to obtain asymptotics—for many examples, see [5, 17].

Recall that $G(z) = \sum_{a=1}^k z^a$, ϕ denotes the unique positive real solution to

$$\frac{1}{\phi^1} + \frac{1}{\phi^2} + \dots + \frac{1}{\phi^k} = 1,$$

and $\sigma = 1/\phi$. Thus $G(\sigma) = 1$, so σ is a singularity of $F(z)$ and hence of all of our generating functions. In fact σ is the dominant singularity of all of our generating functions, and we also have $R' > 1$ for all of our generating functions:

Lemma 4.3. *The only number $z \in \mathbb{C}$ for which $G(z) = 1$ and $|z| \leq 1$ is $z = \sigma$.*

Proof. Since we have assumed $k \geq 2$, we have $\phi > 1$, so $0 < \sigma < 1$. Miles [14] showed that $M(z) = z^k - z^{k-1} - \dots - z - 1$ has one root of modulus greater than one, and $k-1$ (distinct) roots of modulus strictly smaller than one. It is straightforward to verify that $r \in \mathbb{C}$ is a root of $M(z)$ if and only if $1/r$ is a root of $G(z) - 1$, which in turn yields the stated result. \square

Thus, the expansion of our generating function $F(z)$ at $z = \sigma$ will play a role in all of our asymptotic derivations.

Lemma 4.4. *The Laurent expansion of*

$$F(z) = \frac{1}{1 - G(z)}$$

at $z = \sigma$ is

$$F(z) = \frac{-1}{G'(\sigma)(z - \sigma)} + \frac{G''(\sigma)}{2G'(\sigma)^2} + l_1(z - \sigma) + l_2(z - \sigma)^2 + \dots$$

for some (unimportant) complex coefficients l_1, l_2, \dots

Proof. Descartes' rule of signs implies that $z = \sigma$ has multiplicity one as a root of $1 - G(z)$. Thus the desired expansion is

$$\frac{a}{z - \sigma} + b + l_1(z - \sigma) + l_2(z - \sigma)^2 + \dots$$

for some constants a, b, l_1, l_2, \dots . We need to establish the values of a and b . Write $1 - G(z) = (z - \sigma)g(z)$ for some polynomial $g(z)$ of which σ is not a root. First, a is the residue of $1/(1 - G(z))$ at $z = \sigma$, so $a = 1/g(\sigma)$. Note that, since σ is a positive real number, the coefficients of $g(z)$ are all real numbers. To compute a , then, we use the continuity of g along the real line and l'Hospital's rule to obtain

$$g(\sigma) = \lim_{x \rightarrow \sigma} g(x) = \lim_{x \rightarrow \sigma} \frac{g(x)(x - \sigma)}{x - \sigma} = \lim_{x \rightarrow \sigma} \frac{1 - G(x)}{x - \sigma} = \lim_{x \rightarrow \sigma} \frac{-G'(x)}{1} = -G'(\sigma), \quad (4.1)$$

and hence $a = -1/G'(\sigma)$.

Next, we have that

$$\frac{1}{g(z)} = \frac{(z - \sigma)}{(z - \sigma)g(z)} = \frac{(z - \sigma)}{1 - G(z)} = a + b(z - \sigma) + l_1(z - \sigma)^2 + l_2(z - \sigma)^3 + \dots$$

is analytic in a disk centered at $z = \sigma$, so

$$b = \frac{d}{dz} \left(\frac{1}{g(z)} \right) \Big|_{z=\sigma} = -\frac{g'(\sigma)}{g(\sigma)^2}.$$

We note that differentiating the equation $g(x)(x - \sigma) = 1 - G(x)$ and performing a little algebra yields

$$g'(x)(x - \sigma) = -G'(x) - g(x).$$

To compute $g'(\sigma)$ we use the continuity of g' along the real line and l'Hospital's rule again:

$$\begin{aligned} g'(\sigma) &= \lim_{x \rightarrow \sigma} g'(x) = \lim_{x \rightarrow \sigma} \frac{g'(x)(x - \sigma)}{x - \sigma} = \lim_{x \rightarrow \sigma} \frac{-G'(x) - g(x)}{x - \sigma} = \lim_{x \rightarrow \sigma} \frac{-G''(x) - g'(x)}{1} \\ &= -G''(\sigma) - g'(\sigma). \end{aligned}$$

(The use of l'Hospital's rule here is justified by (4.1), which says that $\lim_{x \rightarrow \sigma} -G'(x) - g(x) = -G'(\sigma) - g(\sigma) = 0$.) Finally, solving this equation for $g'(\sigma)$ yields

$$g'(\sigma) = \frac{-G''(\sigma)}{2},$$

and hence

$$b = \frac{G''(\sigma)}{2G'(\sigma)^2},$$

as claimed. \square

Finally, we will also need the straightforward facts that, expanding at the origin, we have

$$\frac{1}{z - \sigma} = - \left(\frac{1}{\sigma} + \frac{z}{\sigma^2} + \frac{z^2}{\sigma^3} + \cdots \right), \quad (4.2)$$

and

$$\frac{1}{(z - \sigma)^2} = \frac{1}{\sigma^2} + \frac{2z}{\sigma^3} + \frac{3z^2}{\sigma^4} + \cdots. \quad (4.3)$$

4.2. Asymptotic formulas. We are now ready to establish our asymptotic formulas. We begin with F_n . The asymptotic for F_n stated here is equivalent to the one in [6], where it was derived using the theory of recurrence relations whose characteristic polynomials have no repeated roots. Here we offer an alternate derivation.

Theorem 4.5. *We have*

$$F_n = \frac{\phi^{n+1}}{G'(\sigma)} + O(A^n)$$

for some number A with $0 < A < 1$.

Proof. By Lemma 4.4, the expansion of the generating function $F(z) = 1/(1 - G(z))$ at its dominant singularity, $z = \sigma$, is

$$-\frac{1}{G'(\sigma)(z - \sigma)} + J(z - \sigma)$$

where $J(z - \sigma)$ consists of $(z - \sigma)^0$ and higher-order terms. The expansion of the principal part of this function,

$$\frac{-1}{G'(\sigma)(z - \sigma)},$$

at the origin is, by Equation (4.2),

$$S(z) = \frac{1}{G'(\sigma)} \left(\frac{1}{\sigma} + \frac{z}{\sigma^2} + \frac{z^2}{\sigma^3} + \cdots \right).$$

Thus

$$[z^n]S(z) = \frac{1}{\sigma^{n+1}G'(\sigma)}$$

and so

$$F_n = [z^n]F(z) = \frac{1}{\sigma^{n+1}G'(\sigma)} + O\left(\left(\frac{1}{R'} + \epsilon\right)^n\right)$$

for any $\epsilon > 0$, where R' is the distance from the origin to the singularity of $F(z)$, other than σ , closest to the origin. Lemma 4.3 implies that $R' > 1$, so $\epsilon > 0$ can be chosen for which $0 < (1/R' + \epsilon) < 1$. Since $\phi = 1/\sigma$, this establishes our result with $A = 1/R' + \epsilon$. \square

Next we turn to T_n .

Theorem 4.6. *We have*

$$T_n = \frac{\phi^{n+2}}{G'(\sigma)^2}(n+1) + \frac{\phi^{n+1}G''(\sigma)}{G'(\sigma)^3} - \frac{\phi^{n+1}}{G'(\sigma)} + O(A^n)$$

for some number A with $0 < A < 1$.

Since $A_n = T_n/F_n$, if one accepts as given the asymptotic for A_n from Theorem 1.1, one can immediately deduce this asymptotic for T_n by multiplying the asymptotic formula for F_n in Theorem 4.5 by the one for A_n in Theorem 1.1. We will also give our own proof of the asymptotic for A_n below which is based on our asymptotic for T_n , so we also offer the following direct derivation of Theorem 4.6.

Proof of Theorem 4.6. By Lemma 4.4, we have that the expansion of the generating function $T(z) = 1/(1 - G(z))^2 - 1/(1 - G(z))$ at its dominant singularity, $z = \sigma$, is

$$\frac{1}{G'(\sigma)^2(z - \sigma)^2} + \frac{G'(\sigma)^2 - G''(\sigma)}{G'(\sigma)^3(z - \sigma)} + J(z - \sigma),$$

where $J(z - \sigma)$ consists of $(z - \sigma)^0$ and higher-order terms. The z^n term of the expansion of the principal part of this function at the origin is, by Equations (4.2) and (4.3),

$$\frac{(n+1)}{G'(\sigma)^2\sigma^{n+2}} + \frac{G''(\sigma) - G'(\sigma)^2}{G'(\sigma)^3\sigma^{n+1}},$$

so

$$T_n = \frac{(n+1)}{G'(\sigma)^2\sigma^{n+2}} + \frac{G''(\sigma) - G'(\sigma)^2}{G'(\sigma)^3\sigma^{n+1}} + O\left(\left(\frac{1}{R'} + \epsilon\right)^n\right)$$

for any $\epsilon > 0$, where R' is the distance from the origin to the singularity of $1/(1 - G(z))$, other than σ , closest to the origin. Again, Lemma 4.3 implies that $R' > 1$, so $\epsilon > 0$ can be chosen for which $0 < (1/R' + \epsilon) < 1$, establishing our result with $A = 1/R' + \epsilon$. \square

Next we turn to A_n , whose asymptotic here was proved in a more general setting in [5, Theorem V.1]. Our proof is similar to theirs and foreshadows our proof of Theorem 4.9.

Theorem 4.7. *We have*

$$A_n = \frac{\phi}{G'(\sigma)}(n+1) - 1 + \frac{G''(\sigma)}{G'(\sigma)^2} + O(A^n)$$

for some number A with $0 < A < 1$.

Proof. From Theorems 4.5 and 4.6 we have

$$F_n = \frac{\phi^{n+1}}{G'(\sigma)} + E(n) = \frac{\phi^{n+1}}{G'(\sigma)} \left(1 + \frac{E(n)G'(\sigma)}{\phi^{n+1}}\right)$$

and

$$T_n = \frac{\phi^{n+2}}{G'(\sigma)^2}(n+1) + \frac{\phi^{n+1}G''(\sigma)}{G'(\sigma)^3} - \frac{\phi^{n+1}}{G'(\sigma)} + E_2(n)$$

for some functions E and E_2 with $E(n) = O(A^n)$ and $E_2(n) = O(A^n)$ for some constant A with $0 < A < 1$. Let B be any constant with $A < B < 1$. Then

$$\begin{aligned} A_n = \frac{T_n}{F_n} &= \left(\frac{\phi}{G'(\sigma)}(n+1) + \frac{G''(\sigma)}{G'(\sigma)^2} - 1 + \frac{E_2(n)G'(\sigma)}{\phi^{n+1}} \right) \left(\frac{1}{1 + E(n)G'(\sigma)/\phi^{n+1}} \right) \\ &= \left(\frac{\phi}{G'(\sigma)}(n+1) + \frac{G''(\sigma)}{G'(\sigma)^2} - 1 + E_3(n) \right) \left(\frac{1}{1 + E_4(n)} \right) \end{aligned}$$

for

$$E_3(n) = \frac{E_2(n)G'(\sigma)}{\phi^{n+1}}$$

and

$$E_4(n) = \frac{E(n)G'(\sigma)}{\phi^{n+1}}.$$

since σ is constant and $\phi > 1$, it is immediate that $E_3(n) = O(A^n)$ and $E_4 = O(A^n)$. It is therefore also straightforward to obtain

$$\frac{1}{1 + E_4(n)} = 1 + O(A^n).$$

Thus, we have

$$A_n = \left(\frac{\phi}{G'(\sigma)}(n+1) + \frac{G''(\sigma)}{G'(\sigma)^2} - 1 + O(A^n) \right) (1 + O(A^n)).$$

Multiplying this out and using $nO(A^n) = O(B^n)$ along with the fact that $G'(\sigma) > 0$, $G''(\sigma) > 0$, and $\phi > 0$ are constants, we obtain

$$A_n = \frac{\phi}{G'(\sigma)}(n+1) - 1 + \frac{G''(\sigma)}{G'(\sigma)^2} + O(B^n)$$

Since $0 < B < 1$, we are done. \square

Next we turn to $C_{n,j}$.

Theorem 4.8. *Let $j \in \{1, 2, \dots, k\}$. Then we have*

$$C_{n,j} = \frac{\phi^{n+2-j}}{G'(\sigma)^2}(n+1-j) + \frac{\phi^{n+1-j}G''(\sigma)}{G'(\sigma)^3} + O(A^n)$$

for some number A with $0 < A < 1$.

Proof. We have

$$C_{n,j} = [x^n] \frac{x^j}{(1 - G(x))^2} = [x^{n-j}] \frac{1}{(1 - G(x))^2}.$$

Lemma 4.4 implies that the expansion of $1/(1 - G(z))^2$ at its dominant singularity, $z = \sigma$, is

$$\frac{1}{G'(\sigma)^2(z - \sigma)^2} - \frac{G''(\sigma)}{G'(\sigma)^3(z - \sigma)} + J(z - \sigma),$$

where $J(z - \sigma)$ consists of $(z - \sigma)^0$ and higher-order terms. The z^n term of the expansion of the principal part of this function at the origin is, by Equations (4.2) and (4.3),

$$\frac{(n+1)}{G'(\sigma)^2\sigma^{n+2}} + \frac{G''(\sigma)}{G'(\sigma)^3\sigma^{n+1}},$$

and thus

$$C_{n,j} = \frac{(n+1-j)}{G'(\sigma)^2\sigma^{n+2-j}} + \frac{G''(\sigma)}{G'(\sigma)^3\sigma^{n+1-j}} + O\left(\left(\frac{1}{R'} + \epsilon\right)^n\right)$$

for any $\epsilon > 0$, where R' is the distance from the origin to the singularity of $1/(1 - G(z))$, other than σ , closest to the origin. Again, Lemma 4.3 implies that $R' > 1$, so $\epsilon > 0$ can be chosen for which $0 < (1/R' + \epsilon) < 1$, establishing our result with $A = 1/R' + \epsilon$. \square

Finally, we handle $A_{n,j}$.

Theorem 4.9. *Let $j \in \{1, 2, \dots, k\}$. Then*

$$A_{n,j} = \frac{\phi^{1-j}}{G'(\sigma)}(n+1-j) + \frac{\phi^{-j}G''(\sigma)}{G'(\sigma)^2} + O(A^n)$$

for some number A with $0 < A < 1$.

Proof. Our proof is similar to the proof of Theorem 4.7. By Theorems 4.5 and 4.8 we have

$$F_n = \frac{\phi^{n+1}}{G'(\sigma)} + E(n) = \frac{\phi^{n+1}}{G'(\sigma)} \left(1 + \frac{E(n)G'(\sigma)}{\phi^{n+1}} \right)$$

and

$$C_{n,j} = \frac{\phi^{n+2-j}}{G'(\sigma)^2}(n+1-j) + \frac{\phi^{n+1-j}G''(\sigma)}{G'(\sigma)^3} + E_2(n)$$

for some functions E and E_2 with $E(n) = O(A^n)$ and $E_2(n) = O(A^n)$ for some constant A with $0 < A < 1$. Let B be any constant with $A < B < 1$. Then

$$\begin{aligned} A_{n,j} = \frac{C_{n,j}}{F_n} &= \left(\frac{\phi^{1-j}}{G'(\sigma)}(n+1-j) + \frac{\phi^{-j}G''(\sigma)}{G'(\sigma)^2} + \frac{E_2(n)G'(\sigma)}{\phi^{n+1}} \right) \left(\frac{1}{1 + E(n)G'(\sigma)/\phi^{n+1}} \right) \\ &= \left(\frac{\phi^{1-j}}{G'(\sigma)}(n+1-j) + \frac{\phi^{-j}G''(\sigma)}{G'(\sigma)^2} + E_3(n) \right) \left(\frac{1}{1 + E_4(n)} \right) \end{aligned}$$

for

$$E_3(n) = \frac{E_2(n)G'(\sigma)}{\phi^{n+1}}$$

and

$$E_4(n) = \frac{E(n)G'(\sigma)}{\phi^{n+1}}.$$

since σ is constant and $\phi > 1$, it is immediate that $E_3(n) = O(A^n)$ and $E_4 = O(A^n)$. It is therefore also straightforward to obtain

$$\frac{1}{1 + E_4(n)} = 1 + O(A^n).$$

Thus, we have

$$A_{n,j} = \left(\frac{\phi^{1-j}}{G'(\sigma)}(n+1-j) + \frac{\phi^{-j}G''(\sigma)}{G'(\sigma)^2} + O(A^n) \right) (1 + O(A^n)).$$

Multiplying this out and using $nO(A^n) = O(B^n)$ along with the fact that $G'(\sigma) > 0$, $G''(\sigma) > 0$, and $\phi > 0$ are constants, we obtain

$$A_{n,j} = \frac{\phi^{1-j}}{G'(\sigma)}(n+1-j) + \frac{\phi^{-j}G''(\sigma)}{G'(\sigma)^2} + O(B^n).$$

Since $0 < B < 1$, we are done. □

TABLE 1. Values of $C_{n,j}$ for $k = 2, 3, 4$

$k = 2$			$k = 3$				$k = 4$				
n	$C_{n,1}$	$C_{n,2}$	n	$C_{n,1}$	$C_{n,2}$	$C_{n,3}$	n	$C_{n,1}$	$C_{n,2}$	$C_{n,3}$	$C_{n,4}$
0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	1	1	0	0	1	1	0	0	0
2	2	1	2	2	1	0	2	2	1	0	0
3	5	2	3	5	2	1	3	5	2	1	0
4	10	5	4	12	5	2	4	12	5	2	1
5	20	10	5	26	12	5	5	28	12	5	2
6	38	20	6	56	26	12	6	62	28	12	5
7	71	38	7	118	56	26	7	136	62	28	12
8	130	71	8	244	118	56	8	294	136	62	28
9	235	130	9	499	244	118	9	628	294	136	62
10	420	235	10	1010	499	244	10	1328	628	294	136
11	744	420	11	2027	1010	499	11	2787	1328	628	294
12	1308	744	12	4040	2027	1010	12	5810	2787	1328	628
13	2285	1308	13	8004	4040	2027	13	12043	5810	2787	1328
14	3970	2285	14	15776	8004	4040	14	24840	12043	5810	2787
15	6865	3970	15	30956	15776	8004	15	51016	24840	12043	5810
16	11822	6865	16	60504	30956	15776	16	104380	51016	24840	12043
17	20284	11822	17	117845	60504	30956	17	212848	104380	51016	24840
18	34690	20284	18	228818	117845	60504	18	432732	212848	104380	51016
19	59155	34690	19	443057	228818	117845	19	877400	432732	212848	104380
20	100610	59155	20	855732	443057	228818	20	1774672	877400	432732	212848

5. TABLES FOR SMALL VALUES OF n AND k

In this section we give tables of values for F_n , T_n , A_n , $C_{n,j}$, and $A_{n,j}$ and our asymptotic approximations to them for small values of n and k .

First, Table 1 gives the values of $C_{n,j}$ for $j \in \{1, 2, \dots, k\}$, for $k = 2, 3, 4$ and $n = 0$ to $n = 20$. As we mentioned after the proof of Theorem 3.2, for a fixed value of k , the sequences $\{C_{n,j}\}_{n=0}^{\infty}$ for $j \in \{1, 2, \dots, k\}$ are all just shifts of the same sequence.

Next, Tables 2, 3, and 4 contain the values of F_n , T_n , A_n , $C_{n,1}$, and $A_{n,1}$ and our asymptotic formula approximations to them for $k = 2, 3, 4$ and $n = 0$ to $n = 15$. Table 2 corresponds to $k = 2$, Table 3 corresponds to $k = 3$, and Table 4 corresponds to $k = 4$. Entries in these tables are rounded to 3 decimal places. In these tables, the approximating formulas we are using are those given in Theorem 1.1 without the $O(A^n)$ terms. So, for instance, our approximation to F_n is given by

$$F_n \approx \frac{\phi^{n+1}}{G'(\sigma)}.$$

We also remark that for $k = 2$, Binet's formula leads to the same approximating asymptotic for F_n —for $n \geq 0$, Binet's formula for the n th Fibonacci number yields

$$F_n = \frac{1}{\sqrt{5}} \left(\phi^{n+1} - \left(\frac{-1}{\phi} \right)^{n+1} \right).$$

TABLE 2. Values of F_n , T_n , A_n , $C_{n,1}$, and $A_{n,1}$ for $k = 2$; $\phi = (1 + \sqrt{5})/2$

n	F_n	Appr.	T_n	Appr.	A_n	Appr.	$C_{n,1}$	Appr.	$A_{n,1}$	Appr.
0	1	0.724	0	0.089	0.0	0.124	0	0.179	0.0	0.247
1	1	1.171	1	0.992	1.0	0.847	1	0.813	1.0	0.694
2	2	1.894	3	2.976	1.5	1.571	2	2.163	1.0	1.142
3	3	3.065	7	7.033	2.333	2.294	5	4.87	1.667	1.589
4	5	4.96	15	14.968	3.0	3.018	10	10.098	2.0	2.036
5	8	8.025	30	30.026	3.75	3.742	20	19.928	2.5	2.483
6	13	12.985	58	57.979	4.462	4.465	38	38.051	2.923	2.93
7	21	21.01	109	109.015	5.19	5.189	71	70.964	3.381	3.378
8	34	33.994	201	200.989	5.912	5.912	130	130.025	3.824	3.825
9	55	55.004	365	365.008	6.636	6.636	235	234.983	4.273	4.272
10	89	88.998	655	654.995	7.36	7.36	420	420.012	4.719	4.719
11	144	144.001	1164	1164.004	8.083	8.083	744	743.992	5.167	5.167
12	233	232.999	2052	2051.997	8.807	8.807	1308	1308.005	5.614	5.614
13	377	377.001	3593	3593.002	9.531	9.53	2285	2284.997	6.061	6.061
14	610	610.0	6255	6254.999	10.254	10.254	3970	3970.002	6.508	6.508
15	987	987.0	10835	10835.001	10.978	10.978	6865	6864.999	6.955	6.955

TABLE 3. Values of F_n , T_n , A_n , $C_{n,1}$, and $A_{n,1}$ for $k = 3$; $\phi \approx 1.8392868$

n	F_n	Appr.	T_n	Appr.	A_n	Appr.	$C_{n,1}$	Appr.	$A_{n,1}$	Appr.
0	1	0.618	0	0.132	0.0	0.213	0	0.2	0.0	0.323
1	1	1.137	1	0.946	1.0	0.832	1	0.75	1.0	0.66
2	2	2.092	3	3.034	1.5	1.45	2	2.083	1.0	0.996
3	4	3.848	8	7.96	2.0	2.069	5	5.126	1.25	1.332
4	7	7.078	19	19.017	2.714	2.687	12	11.808	1.714	1.668
5	13	13.018	43	43.028	3.308	3.305	26	26.095	2.0	2.005
6	24	23.943	94	93.948	3.917	3.924	56	56.046	2.333	2.341
7	44	44.038	200	200.032	4.545	4.542	118	117.891	2.682	2.677
8	81	80.999	418	418.008	5.16	5.161	244	244.07	3.012	3.013
9	149	148.98	861	860.968	5.779	5.779	499	499.007	3.349	3.349
10	274	274.017	1753	1753.025	6.398	6.397	1010	1009.948	3.686	3.686
11	504	503.996	3536	3535.998	7.016	7.016	2027	2027.043	4.022	4.022
12	927	926.994	7077	7076.985	7.634	7.634	4040	4039.994	4.358	4.358
13	1705	1705.007	14071	14071.015	8.253	8.253	8004	8003.979	4.694	4.694
14	3136	3135.997	27820	27819.995	8.871	8.871	15776	15776.023	5.031	5.031
15	5768	5767.998	54736	54735.994	9.49	9.49	30956	30955.993	5.367	5.367

Since $0 < 1/\phi < 1$, this gives the asymptotic formula $F_n \sim \phi^{n+1}/\sqrt{5}$ —specifically, we have

$$F_n = \frac{\phi^{n+1}}{\sqrt{5}} + E(n),$$

where the error term

$$E(n) = -\frac{1}{\sqrt{5}} \left(\frac{-1}{\phi} \right)^{n+1}$$

TABLE 4. Values of F_n , T_n , A_n , $C_{n,1}$, and $A_{n,1}$ for $k = 4$; $\phi \approx 1.92756198$

n	F_n	Appr.	T_n	Appr.	A_n	Appr.	$C_{n,1}$	Appr.	$A_{n,1}$	Appr.
0	1	0.566	0	0.162	0.0	0.287	0	0.212	0.0	0.374
1	1	1.092	1	0.931	1.0	0.853	1	0.729	1.0	0.667
2	2	2.104	3	2.986	1.5	1.419	2	2.023	1.0	0.961
3	4	4.056	8	8.053	2.0	1.986	5	5.091	1.25	1.255
4	8	7.818	20	19.951	2.5	2.552	12	12.11	1.5	1.549
5	15	15.07	47	46.993	3.133	3.118	28	27.77	1.867	1.843
6	29	29.049	107	107.033	3.69	3.685	62	62.063	2.138	2.136
7	56	55.994	238	238.024	4.25	4.251	136	136.082	2.429	2.43
8	108	107.931	520	519.932	4.815	4.817	294	294.017	2.722	2.724
9	208	208.044	1120	1120.024	5.385	5.384	628	627.863	3.019	3.018
10	401	401.017	2386	2386.03	5.95	5.95	1328	1328.068	3.312	3.312
11	773	772.985	5037	5036.995	6.516	6.516	2787	2787.047	3.605	3.606
12	1490	1489.977	10553	10552.957	7.083	7.083	5810	5809.98	3.899	3.899
13	2872	2872.024	21968	21968.029	7.649	7.649	12043	12042.935	4.193	4.193
14	5536	5536.004	45480	45480.014	8.215	8.215	24840	24840.053	4.487	4.487
15	10671	10670.99	93709	93708.986	8.782	8.782	51016	51016.018	4.781	4.781

approaches 0 at an exponential rate. Indeed, this is the same as our asymptotic $F_n \sim \phi^{n+1}/G'(\sigma)$ for $k = 2$, because for $k = 2$ we have $\phi = (1 + \sqrt{5})/2$ and $G'(x) = 1 + 2x$, so $G'(\sigma) = G'(1/\phi) = \sqrt{5}$.

Finally, because all of our asymptotic formulas have absolute error terms that decay exponentially, in Tables 2, 3, and 4 we obtain excellent approximations to our values F_n , T_n , A_n , $C_{n,j}$, and $A_{n,j}$ even for small values of n .

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